Transient excitation of linear space-charge waves by a punctual source in a drifting cold plasma



2006 J. Phys. A: Math. Gen. 39 9937

(http://iopscience.iop.org/0305-4470/39/31/020)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 9937–9960 doi:10.1088/0305-4470/39/31/020

# Transient excitation of linear space-charge waves by a punctual source in a drifting cold plasma

**Orélien C Randriamboarison**

Université d'Orléans and Laboratoire de Physique et Chimie de l'Environnement,
UMR 6115 du Centre National de la Recherche Scientifique, 45071 Orléans Cédex 02, France

E-mail: randriam@cnrs-orleans.fr

Received 3 March 2006, in final form 21 June 2006
Published 19 July 2006
Online at stacks.iop.org/JPhysA/39/9937

**Abstract**
We study the effects of a uniform drift on the excitation emanating from an external antenna immersed in a cold modelled plasma. Instead of the well-known approach based on the steady-state regime approximation, the causal and transient evolution of the excited plasma waves is investigated. A comprehensive description of the computational method is proposed. We show that a technique based on the Heaviside direct operational method turns out to be very effective in order to handle the integrodifferential expression for the solution. Making use of a pure analytical calculation, an exact algebraic expression for the plasma response is inferred. The solution takes the form of a combination of Bessel's functions and series of Lommel's functions. Then some specific analysis of the result allow us to gain more understanding on the dynamics of the fast and slow space-charge waves. A special attention is paid to the analysis of the extended singularity of the wave field and the secular behaviour of the fast wave mode at the resonant excitation.

PACS numbers: 52.35.Fp, 52.40.Fp, 52.25.−b, 02.30.Jr

## 1. Introduction

An important plasma property is the stability of its macroscopic space-charge neutrality. When a plasma is instantaneously disturbed from the equilibrium condition, the resulting space-charge fields give rise to collective particle motions that tend to restore the original charge neutrality. These collective motions are characterized by a natural oscillation at the plasma frequency $\omega_p$. Independent of the wave vector, the energy of that pure electrostatic oscillation does not propagate away from its point of origin and the small perturbation of the plasma remains stationary fluctuations.

In a drifting plasma, the dispersion law of the electric mode of oscillations changes and the so-called space-charge disturbances will propagate from one place to another. Indeed, if the charged particles are moving with some uniform velocity, their density gradient associated with any space-charge perturbation will be transported bodily at this speed. The propagation characteristics of the disturbances on drifting electron plasma have been discussed in detail in standard textbooks and monographs in plasma physics. For steady-state harmonic time-dependent disturbances, there were two waves (usually called the fast and slow space-charge waves) given by the dispersion relation $(\omega - kV) = \pm\omega_p$. The wave number is $k$, $\omega$ the frequency of the disturbance, and $V$ the average drift velocity of the electrons. These waves constitute electrostatic or longitudinal modes, the current they produce exactly cancels out Maxwell's displacement current, so (1) the wave-vector is parallel to the electric field, (2) the disturbances have no magnetic field component and (3) the curl free electric field of each mode can be derived from an electric scalar potential.

Wave excitation in a three-dimensional drifting plasma is an essential aspect of the laboratory and space plasma physics. Owing to its fundamental nature and its importance in applications, the problem already received salient effort during the seventies. A paper by Chassériaux (1971) has been specifically devoted to the study of the effect of the plasma drift on the space-charge waves emanating from a point source immersed in a fluid plasma. Moreover, the problem under consideration may also be regarded as the cold plasma response associated with more general descriptions given by Fiala (1973), Michel (1976), Mourgues *et al* (1980) and others. Their results have been successful in interpreting space plasma physics measurements and they will continue to find widespread use in most circumstances.

In the previous publications on the topic, however, the transient effects were ignored. The description usually concerned the harmonic or modal response of the plasma at a fixed frequency during the permanent state regime that is remote in time from the switch-on of the exciter. In this paper we attempt to investigate the transient excitation and dynamical propagation of space-charge electrostatic waves within a drifting plasma. The response of the system is desired at a given time $t$ for an excitation beginning at time zero. It is worth noting that in the general case where a dissipative plasma is considered, unless the collision frequency is very small, no permanent regime may be characterized in the time course of the excited waves. Recourse to an exact treatment of their dynamics seems thus unavoidable.

A switched-on evolution of the perturbation of plasma was handled by Chee-Seng (1985), but principally the results he proposed consist of some general integral representations of the solution. Here, we aim to develop a method which provides an analytically tractable solution of the underlying problem. To this end and complying with Chassériaux's (1971) initial model, a cold, fluid and drifting plasma is considered. A comprehensive description of the computation method will be given. The technique we opt for differs from the classical investigation on transient phenomenon problems. Instead of the well-known framework of Laplace–Fourier transformation, a method essentially based on the Heaviside direct operational theory is presented. This provides us with the exact algebraic expression for the propagating space-charge waves.

Except for very limited physical situations, the cold modelled plasma does not provide but a very simplified description for the problem. Hence, the focus of the present investigation resides in the theoretical formulation. Actual physical applications require more developed or sophistical plasma models which will be the subject of some subsequent works.

The paper is organized as follows. Section 2 gives the fundamental assumptions we consider and the propagation equation we may derive in rapport with them. Section 3 focuses on the description of the technique used to separate the independent variables in order to simplify the derivation of the solution. Note that the essential difficulty in dealing with the propagation problem in a moving material medium comes from the fact that the time and space variables mix together in the expression of the solution integral. Often, the separation of these variables makes the calculation more tractable. A detailed sketch of the method of solution is then given in section 4. Explicit expressions of the solution in some specific situations are proffered in section 5. The results which concern the drift plasma response are collected along with the corresponding discussion throughout section 6. The opportunity will be taken to highlight the main characteristics of the excited waves. Namely, special attention is paid to the resonant excitation. A summary and conclusions form the last section.

## 2. Basic assumptions and equations

We consider the problem of the excitation by a yet arbitrary time-dependent point source in a uniformly moving, homogeneous, infinite and cold unmagnetized plasma. The charge distribution source is supposed to be localized at the origin $O$ of the laboratory rest system of reference $\mathcal{K}(O; x, y, z)$. The plasma moves with a constant mean velocity $\mathbf{V}$ measured with respect to $\mathcal{K}(O; x, y, z)$. Let $\mathcal{K}'(O'; x', y', z')$ denote the drifting Cartesian frame of reference associated with the medium. Without loss of generality, we match the origins $O$ and $O'$ of these frames at the initial instant $t = 0$. The relation between the coordinates and time used by the respective observers attached with $\mathcal{K}$ and $\mathcal{K}'$ for the description of motions is the pure Galilean transformation

$$\begin{cases} t = t' \\ \mathbf{r} = \mathbf{r}' + \mathbf{V}t'. \end{cases} \tag{1}$$

We denote quantities measured in the plasma rest frame by a prime ($'$). Inertial frames in nature are related precisely by a Lorentz transformation; a Galilean transformation approximates a Lorentz transformation for $(\|\mathbf{V}\|\, c^{-1}) \ll 1$, $c$ being the speed of light in free space. In other words, the non-relativistic (in Einstein's sense) physics, Newton's laws and Poisson's equation, considered here, are applicable to motion involving material velocities small in absolute magnitude compared to the speed of light, and it is postulated that the basic laws of non-relativistic physics take the same form in all inertial frames of reference. Consequently, in the rest system $\mathcal{K}'$ of the plasma, the continuity equation in linearized approximation reads

$$\frac{\partial N'(\mathbf{r}', t')}{\partial t'} + n_0 \nabla' \cdot \mathbf{u}'(\mathbf{r}', t') = 0, \tag{2}$$

where $n_0$ designates the equilibrium electron (and ion) number density, $N'$ and $\mathbf{u}'$ represent the perturbations of the number density and average velocity, respectively. To find the average velocity $\mathbf{u}'$ for the electrons in the fluid plasma when there is an electrostatic potential $\phi(\mathbf{r}', t')$ acting, we use the Langevin equation

$$m\left(\nu' + \frac{\partial}{\partial t'}\right)\mathbf{u}'(\mathbf{r}', t') = e\nabla'\phi(\mathbf{r}', t') \tag{3}$$

where $m$ is the electron mass, $e$ is the magnitude of the elementary charge, $\nu'$ is the phenomenological collision frequency and represents the number of collisions per second which the average electron has with heavy particles in the plasma. Equations (2) and (3) hold from the principle of the invariance of the equations of fluid mechanics under the Galilean transformation. Note that, rigorously, $\nu' = \nu(1 - \|\mathbf{V}\|^2 c^{-2})^{-1/2}$ (Chawla and Unz 1962, Unz 1966), where $\nu$ denotes the improper collision frequency that is measured in the $\mathcal{K}$ frame of reference. In what follows, however, we assume the apparent equality $\nu' \approx \nu$.

The quasi-static approximation of the electrodynamic theory is considered. So, the perturbed number density $N'$ is related to the electric potentials by Poisson's equation,

$$N'(\mathbf{r}', t') = \frac{e}{\varepsilon_0}\nabla'^2[\phi(\mathbf{r}', t') - \phi_{\text{ext}}(\mathbf{r}', t')], \tag{4}$$

where $\phi_{\text{ext}}(\mathbf{r}', t')$ represents an externally applied potential and $\varepsilon_0$ is the free space permittivity. The quasi-static formulation turns out to be all the more suitable since the full set of Maxwell's equations and Galilean relativity, imposed in our treatment, prove to be inconsistent with each other (Vaidya and Farina 1991). Here, we are merely concerned with that which is termed as the electric limit of Galilean electromagnetism fully described by Le Bellac and Lévy-Leblond (1973). We note that a more recent paper by de Montigny and Rousseaux (2006) refines the theory for this specific electrodynamics of moving bodies at low velocities.

Taking the time derivative of (2) and the divergence of (3), the vector velocity $\mathbf{u}'$ may be eliminated between the resulting equations. Then, we obtain the partial differential equation

$$\left(\frac{\partial^2}{\partial t'^2} + \nu\frac{\partial}{\partial t'} + \omega_p^2\right)\nabla'^2[\phi(\mathbf{r}', t') - \phi_{\text{ext}}(\mathbf{r}', t')] = \frac{\omega_p^2}{\varepsilon_0} Q_{\text{ext}}(\mathbf{r}', t') \tag{5}$$

where $\omega_p^2 = (n_0 e^2/\varepsilon_0 m)$, $\omega_p$ is the plasma pulsation; $\omega_p' = \omega_p$ (Chawla and Unz 1962, Unz 1966). The extraneous charge distribution $Q_{\text{ext}}(\mathbf{r}', t')$ such as

$$\nabla'^2\phi_{\text{ext}}(\mathbf{r}', t') + Q_{\text{ext}}(\mathbf{r}', t')/\varepsilon_0 = 0 \tag{6}$$

has been introduced in (5).

If we adopt a new dependent variable defined by

$$\chi(\mathbf{r}', t') = \exp(\nu t'/2)[\phi(\mathbf{r}', t') - \phi_{\text{ext}}(\mathbf{r}', t')], \tag{7}$$

the governing equation (5) reduces to

$$\left(\frac{\partial^2}{\partial t'^2} + \Omega^2\right)\nabla'^2\chi(\mathbf{r}', t') = S(\mathbf{r}', t'). \tag{8}$$

Here, $\Omega$ is defined by the relation $\Omega^2 = \left(\omega_p^2 - \nu^2/4\right)$. It is assumed that $\omega_p > (\nu/2)$. The free term (or source) on the right-hand side of (8) now reads

$$S(\mathbf{r}', t') = \left(\omega_p^2/\varepsilon_0\right) \mathrm{e}^{\nu t'/2} Q_{\text{ext}}(\mathbf{r}', t'). \tag{9}$$

Here, we are concerned with a time-periodic and causal punctual source antenna localized at the origin $O$ of the laboratory frame of reference. Then, the charge distribution in this frame has the form

$$Q_{\text{ext}}(\mathbf{r}, t) = q_0\delta_3(\mathbf{r})H(t)\sin(\omega_0 t), \tag{10}$$

where $q_0$ is a constant charge (in Coulomb per $\text{m}^3$), $\omega_0$ is the external impressed pulsation. $H$ represents the Heaviside unit step function and $\delta_3$ the Dirac delta function in the three space dimensions. According to Poisson's equation (6), the related external electric potential is expressed by

$$\phi_{\text{ext}}(\mathbf{r}, t) = q_0 H(t)\sin(\omega_0 t)/(4\pi\varepsilon_0\|\mathbf{r}\|). \tag{11}$$

Here and in what follows, the symbol $\|\cdot\|$ designates the scalar magnitude of the vectorial quantity throughout. Substituting (10) into the source term expression (9), and transforming the result as a function of the primed coordinates and time, we have

$$S(\mathbf{r}', t') = (q_0/\varepsilon_0)\omega_p^2\delta_3(\mathbf{r}' + \mathbf{V}t')H(t')\,\mathrm{e}^{\nu t'/2}\sin(\omega_0 t). \tag{12}$$

It is seen that the differential equation is a rather simple one when we consider its form in the rest frame $\mathcal{K}'$ of the plasma. The source term in this frame however is more complicated since time and space variables mix in the argument of the Dirac delta function, as seen on the right-hand side of (12). In general, such a point may lead to an arduousness in the derivation of the solution. Here, recourse to the technique of the direct operational method is made to overcome the difficulty. For instance, we shall make use of the easily derivable (Arfken 1970, for example) formula

$$\exp(t'\mathbf{V}\cdot\nabla')\delta_3(\mathbf{r}') = \delta_3(\mathbf{r}' + \mathbf{V}t') \tag{13}$$

to rearrange expression (12) in order to symbolically separate the spatial and temporal variables. A drifting operator may then be used to embody the total effect of the translatory motion of the plasma medium. The investigation is first performed in the frame $\mathcal{K}'$ . The passage into the other frame, namely, the laboratory frame of reference, will be made after the expression of the solution is obtained.

In addition, we note that the source term $S$ and the dependant variable $\chi$ transform as scalar fields, so we have directly $S(\mathbf{r}', t') = S[\mathbf{r}'(\mathbf{r},t), t'(\mathbf{r},t)]$ and $\chi(\mathbf{r}', t') = \chi[\mathbf{r}'(\mathbf{r},t), t'(\mathbf{r},t)]$. This is the reason why primed representations have not been defined for these quantities.

## 3. Integrodifferential expression of the solution

First, we observe that (8) represents the governing equation of the simple space-charge oscillations in the framework of a cold stationary plasma. It is straightforward to show that the functional

$$G(\mathbf{r}', t') = -\frac{1}{4\pi\|\mathbf{r}'\|}\frac{H(t')}{\Omega}\sin(\Omega t') \tag{14}$$

solves the Green equation

$$\left(\frac{\partial^2}{\partial t'^2} + \Omega^2\right)\nabla'^2 G(\mathbf{r}', t') = \delta_3(\mathbf{r}')\delta(t'). \tag{15}$$

By the superposition principle, the causal solution of (8) is derived as the convolution with respect to space and time of the Green function (14) and the source term (12). That is,

$$\chi(\mathbf{r}', t') = -\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p^2}{\Omega}H(t')\int_0^{t'}\frac{\mathrm{e}^{\nu t_1/2}\sin(\omega_0 t_1)}{\|\mathbf{r}' + \mathbf{V}t_1\|}\sin[\Omega(t' - t_1)]\,\mathrm{d}t_1. \tag{16}$$

where the integration with respect to space variables has been immediately carried out, since it simply consists of a convolution with a Dirac delta function.

To progress, we now infer the expression of the solution in terms of the time and space variables of the laboratory frame of reference, $\mathcal{K}$. In general, the difficulty when performing the integration in equation (16) or in its transformed representation which describes the response of a moving medium lies in the fact that position and time variables mix together. This is seen at the denominator of the integrand of equation (16). Such a situation may be bypassed when we proceed as follows. First, we note that the relations concerning the differentiation with respect to time and space variables in both frames of reference are given by

$$\nabla' = \nabla, \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla. \tag{17}$$

Within the framework of the Galilean transformations, the differential operators with respect to the space variables are invariant under the change from one set of coordinates to another. Second, from the above-mentioned direct operational rule and according to (1) and (17) one may write

$$\begin{aligned}\frac{1}{\|\mathbf{r}' + \mathbf{V}t_1\|} &= \frac{1}{\|(\mathbf{r} - \mathbf{V}t) + \mathbf{V}t_1\|}\\ &= \exp[-(t - t_1)\mathbf{V}\cdot\nabla]\frac{1}{\|\mathbf{r}\|}.\end{aligned} \tag{18}$$

As a consequence, if (18) is substituted in (16) the time course of the solution reads

$$\chi(\mathbf{r}, t) = -\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p^2}{\Omega}H(t)\int_0^t \mathrm{d}t_1\, \mathrm{e}^{\nu t_1/2}\sin(\omega_0 t_1)\sin[\Omega(t-t_1)]\,\mathrm{e}^{-(t-t_1)\mathbf{V}\cdot\nabla}\frac{1}{\|\mathbf{r}\|}. \tag{19}$$

Equation (19) constitutes the inferred integrodifferential representation of the wave solution. It is fully expressed in terms of the $\mathcal{K}$ reference coordinates. In addition, the required separation of both time and space variables is thereby established in the expression of the integrand. Recourse to a direct operational calculation is naturally made in the following.

Now, we expand the product of trigonometric functions under the integral sign in (19) as a summation. Plugging this expansion into (19), the solution takes the equivalent representation given by

$$\chi(\mathbf{r}, t) = -\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p^2}{\Omega}H(t)\frac{\mathrm{e}^{\nu t/2}}{2}\{\cos(\omega_0 t)[F_R(\omega_+, \mathbf{r}, t) - F_R(\omega_-, \mathbf{r}, t)] + \sin(\omega_0 t)[F_I(\omega_+, \mathbf{r}, t) - F_I(\omega_-, \mathbf{r}, t)]\} \tag{20}$$

where we denote $\omega_\pm = (\omega_0 \pm \Omega)$. The following functional forms have been introduced in (20):

$$F_R(\omega, \mathbf{r}, t) = \int_0^t \mathrm{d}t_1\, \mathrm{e}^{-\nu t_1/2}\cos(\omega t_1)\exp(-t_1\mathbf{V}\cdot\nabla)\frac{1}{\|\mathbf{r}\|} \tag{21}$$

and

$$F_I(\omega, \mathbf{r}, t) = \int_0^t \mathrm{d}t_1\, \mathrm{e}^{-\nu t_1/2}\sin(\omega t_1)\exp(-t_1\mathbf{V}\cdot\nabla)\frac{1}{\|\mathbf{r}\|}. \tag{22}$$

The expression of the total response (20) involves two distinct wave components: the first one oscillates at the characteristic frequency $\omega_+ = (\omega_0 + \Omega)$, whereas the second one varies periodically in time via a lower frequency, $\omega_- = (\omega_0 - \Omega)$. Earlier studies on the dispersion relation of waves in a drifting plasma point out that the higher frequency wave turns out to be the *slow space-charge wave*. The second is associated with the *fast space-charge wave*.

## 4. Method of solution

The underlying problem exhibits an axial symmetry along the direction of the drifting velocity, $\mathbf{V}$, of the infinite plasma. Without any loss of generality, we may choose a cylindrical system of coordinates with a polar axis along this direction. Then, for any observation point specified by the position vector $\mathbf{r}$, let $\rho$ and $z$ be the radial and axial coordinates, respectively: that is, $\|\mathbf{r}\| = (\rho^2 + z^2)^{1/2}$.

Moreover, in order not to have to carry two separate calculations it is convenient to set out the complex functional $F$ defined by

$$F(\omega; \rho, z, t) = F_R(\omega; \rho, z, t) + \mathrm{i}F_I(\omega; \rho, z, t), \tag{23}$$

with the notation $\mathrm{i} = (-1)^{1/2}$. It is straightforward to show that the solution will be determined from the formula

$$\chi(\mathbf{r}, t) = -\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p^2}{\Omega}\frac{H(t)}{2}\mathrm{Re}\{[F(\omega_+; \rho, z, t) - F(\omega_-; \rho, z, t)]\exp[(\nu/2 - \mathrm{i}\omega_0)t]\}, \tag{24}$$

where

$$F(\omega; \rho, z, t) = \int_0^t \mathrm{d}t_1 \exp\left[t_1\left(w - V\frac{\partial}{\partial z}\right)\right]\frac{1}{\|\mathbf{r}\|}, \tag{25}$$

the parameter $w = (\mathrm{i}\omega - \nu/2)$ and $V$ is the module of the plasma drift vector velocity. The notation 'Re' stands for the real part of the complex quantity. Note that $\nu$ is real and positive, whereas the first argument $\omega$ of the functional $F$ may be real positive or negative. The representation (25) is the most convenient form of the solution integral if one of the space variables $\rho$ or $z$ is taken to be zero. Hereafter recourse to the analytical quadrature of this integral with respect to the time will be effected for the investigation of some particular limits of the solution. The following method has proved suitable however in order to evaluate the integral in the general situation.

If we operate throughout (25) with the linear operator $(V\partial/\partial z - w)$, it holds that

$$\left(V\frac{\partial}{\partial z} - w\right)F = -\int_0^t \mathrm{d}\left[\left(w - V\frac{\partial}{\partial z}\right)t_1\right]\exp\left[\left(w - V\frac{\partial}{\partial z}\right)t_1\right]\frac{1}{\|\mathbf{r}\|}$$
$$= \frac{1}{\|\mathbf{r}\|} - \mathrm{e}^{wt}\exp\left(-tV\frac{\partial}{\partial z}\right)\frac{1}{\|\mathbf{r}\|}$$
$$= \frac{1}{\|\mathbf{r}\|} - \frac{\mathrm{e}^{wt}}{\|\mathbf{r}-tV\hat{\mathbf{z}}\|}. \tag{26}$$

Equation (26) proves to be valid everywhere in the plasma volume except at points localized by $\|\mathbf{r}\| = 0$ and $\|\mathbf{r}-tV\hat{\mathbf{z}}\| = 0$, for any time $t$ positive. The continuity of the distance $\|\mathbf{r}\|$ as a function of the $z$ variable is also assumed at any location position except at these two singular points belonging to the $z$-axis. The foregoing equation implies that $F(\omega; \rho, z, t)$ solves the non-homogeneous, pseudo-initial value, linear differential equation

$$\frac{\partial F}{\partial z} - \alpha F = \frac{1}{V}\left[1 - \mathrm{e}^{wt}\exp\left(-tV\frac{\partial}{\partial z}\right)\right]\frac{1}{(\rho^2 + z^2)^{1/2}}, \tag{27}$$

with the notation $\alpha = w/V$. By means of a direct operational inversion method (Kaplan 1967, Lindell 2000), we may write the solution of (27) as

$$F(\omega; \rho, z, t) = [F(\omega; \rho, z=0, t) + V^{-1}\mathcal{I}(-Vt)]\,\mathrm{e}^{\alpha z} + V^{-1}[1 - \mathrm{e}^{wt}\exp(-tV\partial/\partial z)]\,\mathrm{e}^{\alpha z}\mathcal{I}(z) \tag{28}$$

with

$$\mathcal{I}(z) = \int_0^z \frac{\exp(-\alpha\zeta)}{(\rho^2 + \zeta^2)^{1/2}}\,\mathrm{d}\zeta, \tag{29}$$

provided that $\rho$ is strictly positive. In (28), the boundary value condition has been considered from the response at $z = 0$. We emphasize that the solution at this space plane which contains the point source of excitation may be calculated via a parallel or independent way. But here, using the initial definition (25) and in view of (29), it is obviously established that

$$F(\omega; \rho, z = 0, t) \equiv \int_0^t \frac{\mathrm{e}^{wt_1}}{\left(\rho^2 + V^2t_1^2\right)^{1/2}}\,\mathrm{d}t_1$$
$$= -V^{-1}\mathcal{I}(-Vt). \tag{30}$$

So the first term on the right-hand side of (28) equals zero. As a consequence, an algebraic expression of the drifting plasma response may directly be inferred through the analytical quadrature of the integral $\mathcal{I}(z)$.

As it stands, $\mathcal{I}(z)$ may be considered as the generalization of the Laplace transform in the sense of Dunn (1967). Such an integral in general is handled by use of the linear shift property of one sided Laplace transform (Roberts and Kaufman 1966, p 3). In the following, we adopt a more convenient way which preserves us from the evaluation of successive derivatives of the integrand function.

By series expanding the exponential involving $\alpha$ in (29) and inverting the integral and summation signs, $\mathcal{I}(z)$ may be put in the form of

$$\mathcal{I}(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \alpha^k}{k!} A_k(z), \tag{31}$$

where the remaining integral, $A_k(z)$, represents a tabulated one (Gröbner and Hofreiter 1965, p 45). Its analytical expression depends upon the parity of the index $k$ and it is shown (appendix A) that for $j = 0, 1, 2, \ldots$

$$A_{2j}(z) = (-1)^j \frac{\left(\frac{1}{2}\right)_j}{(1)_j} \rho^{2j} \ln[z/\rho + (1 + z^2/\rho^2)^{1/2}] + \frac{1}{2}(\rho^2 + z^2)^{1/2} \times \sum_{\mu=0}^{j-1} (-1)^\mu \frac{\left(j - \mu + \frac{1}{2}\right)_\mu}{(j - \mu)_{\mu+1}} \rho^{2\mu} z^{2j-2\mu-1} \tag{32}$$

and

$$A_{2j+1}(z) = (-1)^{j+1} \frac{(1)_j}{\left(\frac{3}{2}\right)_j} \rho^{2j+1} + \frac{1}{2}(\rho^2 + z^2)^{1/2} \sum_{\mu=0}^{j} (-1)^\mu \frac{(j - \mu + 1)_\mu}{\left(j - \mu + \frac{1}{2}\right)_{\mu+1}} \rho^{2\mu} z^{2j-2\mu}. \tag{33}$$

Here, the notation $(a)_\mu$ designates Pochhammer's symbol. In view of the different terms that compose the sequence $A_k(z)$, the initial integral $\mathcal{I}(z)$ may be seen as the sum of three functions.

First, we consider the contribution of the first term on the right-hand side of (33). This term and consequently its contribution, noted $G_0(\rho)$, to the series (31) do not depend on the variable $z$. The summation concerns only odd integer values of $k$. By use of the relation $(2j + 1)! = (1)_{2j+1} = 4^j (1)_j \left(\frac{3}{2}\right)_j$, we may write

$$G_0(\rho) = (\alpha\rho) \sum_{j=0}^{+\infty} \frac{1}{\left(\frac{3}{2}\right)_j \left(\frac{3}{2}\right)_j} \left(-\frac{\alpha^2 \rho^2}{4}\right)^j = \frac{\pi}{2} \mathbf{H}_0(\alpha\rho), \tag{34}$$

where $\mathbf{H}_0(\zeta)$ denotes the Struve function of order 0 and argument $\zeta$ (Abramowitz and Stegun 1970).

Second, we regard the term which involves the logarithm function on the right-hand side of (32). Now, the summation concerns only even values of the $k$ index of the series (31). They supply to the underlying integral the quantity given by

$$G_1(\rho, z) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{(2j)!} \frac{\left(\frac{1}{2}\right)_j}{(1)_j} (\alpha\rho)^{2j} \ln[z/\rho + (1 + z^2/\rho^2)^{1/2}] = J_0(\alpha\rho) \ln[z/\rho + (1 + z^2/\rho^2)^{1/2}], \tag{35}$$

the second equality stemming from the identity $(2j)! = (1)_{2j} = 4^j \left(\frac{1}{2}\right)_j (1)_j$ and the resulting series being identified as $J_0$, the cylindrical Bessel function of order 0.

Then, to complete the expression of the solution we collect the remaining summands together. They form a combination of two distinct double series, having as prefactor the distance $(\rho^2 + z^2)^{1/2}$. We thus define

$$G_2(\rho, z) = \frac{1}{2}(1 + z^2/\rho^2)^{1/2} \left\{ \sum_{j=0}^{+\infty} \frac{\alpha^{2(j+1)}}{4^{j+1}\left(\frac{1}{2}\right)_{j+1}(1)_{j+1}} \sum_{\mu=0}^{j} (-1)^\mu \frac{\left(j - \mu + \frac{3}{2}\right)_\mu}{(j - \mu + 1)_{\mu+1}} \rho^{2\mu} z^{2j-2\mu+1} \right.$$
$$\left. - \sum_{j=0}^{+\infty} \frac{\alpha^{2(j+1)}}{4^j (1)_j \left(\frac{3}{2}\right)_j} \sum_{\mu=0}^{j} (-1)^\mu \frac{(j - \mu + 1)_\mu}{\left(j - \mu + \frac{1}{2}\right)_{\mu+1}} \rho^{2\mu} z^{2j-2\mu} \right\}. \tag{36}$$

As far as the resulting series remain convergent, both component terms of $G_2(\rho, z)$ may be transformed into double infinite series by use of the formula

$$\sum_{j=0}^{+\infty} \sum_{\mu=0}^{j} u_{\mu,j} = \sum_{j=0}^{+\infty} \sum_{\mu=0}^{+\infty} u_{\mu,j+\mu}. \tag{37}$$

After some elementary transformations and rearrangements which principally concern Pochhammer's symbols (Spanier and Oldham 1987), we realize that the infinite series with respect to the index $\mu$ consist of Lommel's functions. These functions are related with the generalized hypergeometric series by the formula (Magnus *et al* 1966)

$$s_{m,\ell}(x) = \frac{x^{m+1}}{(m+1)^2 - \ell^2}\, {}_1F_2\left(1; \frac{m - \ell + 3}{2}, \frac{m + \ell + 3}{2}; -\frac{x^2}{4}\right). \tag{38}$$

Hence, a reduced expression for (36) holds, and it takes the form of

$$G_2(\rho, z) = -(1 + z^2/\rho^2)^{1/2} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \left(\frac{z}{\rho}\right)^k s_{k,0}(\alpha\rho). \tag{39}$$

This achieves the derivation of the integral (29) and from now on its algebraic expression eventually takes the form of the summation $\mathcal{I}(z) = (G_0 + G_1 + G_2)$.

As $G_0$ does not explicitly depend upon the $z$ coordinate, it is straightforward to check that

$$[1 - \mathrm{e}^{wt} \exp(-tV\partial/\partial z)]\, \mathrm{e}^{\alpha z} G_0(\rho) = 0 \tag{40}$$

and the contribution of (34) to the solution vanishes. Hence, as a result we obtain

$$F(\omega; \rho, z, t) = V^{-1} \mathrm{e}^{\alpha z}[f(\omega; \rho, z) - f(\omega; \rho, z - Vt)] \tag{41}$$

with

$$f(\omega; \rho, z) = G_1(\rho, z) + G_2(\rho, z), \tag{42}$$

the expression of $G_1(\rho, z)$ and $G_2(\rho, z)$ being given by (35) and (39), respectively. Because of the importance of the function $f(\omega; \rho, z)$ in all the subsequent analysis, the leading-order of series expansions and asymptotic forms of this function are derived in appendix B.

For the usual mature dispersion relation approach of Chassériaux (1971), Mourgues *et al* (1980) and others, the amplitudes of the space-charge waves have been invariably computed using an ultimate numerical quadrature of some implicit integrals. It is emphasized that such a numerical computation may also be performed in our transient excitation formulation. In this case, either the initial integral $\mathcal{I}(z)$ of equation (29) or the more developed one arising from the use of $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ defined by (B.10) and (B.11) of appendix B should be considered. It proves that the evaluation is simpler here since each of these integrals does not require but a finite range numerical integration. Moreover, as our investigation concerns only electrostatic waves the endpoint $z$ of (29) or $(\alpha\rho)$ of (B.10) and (B.11) rarely exceeds a few plasma characteristic lengths.

In addition to its integral representations, its explicit algebraic expression has also been derived here. Recourse to a developed numerical algorithm is essential in order to compute the function $G_2(\rho, z)$. From (39), its algebraic form consists of a series of coefficients times functions that satisfy recurrence relations. Such a calculation is very common in computational physics. Press *et al* (1992) and Ng (1968) recommend the Clenshaw algorithm to perform the summation of the series.

## 5. Some limits of the solution

First, the purpose is to obtain the limit for the complex function $F(\omega; \rho, 0, t)$ at any point situated in this separatrix. This describes the evolution of the solution through all transverse directions. Second, investigation of the solution along the $z$-axis will be proposed. A particular attention will be confined to the leading-order behaviour of the solution amplitude close to the linear wake left by the antenna through the drifting plasma. Finally, a word about the large $t$ asymptotic approximation of (42) would be naturally in order.

### *5.1. Expression of the solution at the plane $z = 0$*

The plane defined by $(z = 0)$ represents the separatrix between two distinct regions: a downstream region of detection and an upstream region of detection. Oriented perpendicularly with the direction of the plasma drift, this plane contains the excitation source point. The space-charge perturbations should exhibit an isotropic character within this plane. As the plasma moves in a normal direction, the induced fluctuations turn to be stationary and do not propagate along the transverse directions.

According to equation (30), we readily find

$$F(\omega; \rho, z = 0, t) = -\frac{\pi}{2} V^{-1} \mathbf{H}_0(\alpha\rho) + V^{-1} \left\{ J_0(\alpha\rho) \ln[Vt/\rho + (1 + V^2t^2/\rho^2)^{1/2}] + (1 + V^2t^2/\rho^2)^{1/2} \sum_{k=0}^{+\infty} \frac{1}{k!} (Vt/\rho)^k s_{k,0}(\alpha\rho) \right\}, \tag{43}$$

where $\mathbf{H}_0$ denotes the Struve function of order 0 (Abramowitz and Stegun 1970). We emphasize that the expression in (43) behaves like $Y_0(\alpha\rho)$, the Neumann function of order 0 at a vanishing value of $\rho$. The amplitude of each wave modes then exhibits a logarithmic singularity at this limit. It differs from the stationary plasma response which evolves as $r^{-1}$, a source-like singularity.

Incidentally, we note that the general solution of the problem owes its propagation property to the occurrence of the explicit $(z - Vt)$ dependence through the second term on the right-hand side of (41). This means that the space behaviour of the solution in the plane $z = z_0$ at a fixed time $t = t_0$ may be inferred from that of the plane $z = 0$ at the earlier time $t = (t_0 - z_0/V)$. The solution for all $z$ positive may be deduced by making use of its expression at the transverse directions (43). Especially, if the plasma response is singular at the vicinity of the origin $O$, such a property is transported by the drift motion of the plasma. The singularity then affects all points close to the $z$-axis along the forward direction. We point out that the formation of such a line singularity also finds its justification from the mathematical theory of first order partial differential equation. In the $(z, t)$ plane the line $z = Vt$ forms a characteristic base curve of equation (26). The singularity in the initial data propagates along the characteristics.

### *5.2. Expression of the solution along the z-axis*

Setting the radial space coordinate $\rho$ to be zero but $z \neq 0$, we may calculate the limiting form of the studied function at any point localized along the $z$-axis. As far as the integral has a sense, if $\rho$ tends to zero, we may write

$$F(\omega; \rho = 0, z, t) = \int_0^t \frac{e^{wt_1}}{|z - Vt_1|}\, dt_1. \tag{44}$$

The absolute value entering the denominator of the integrand stems from the fact that the quantity throughout represents a distance. Furthermore, caution is needed here for the integral in (44) diverges under a condition when this aforementioned distance vanishes through the interval of integration. To go further in the development, two distinct cases must be considered: the evolution at the upstream region of detection and the evolution at the downstream region of detection.

*5.2.1. Upstream region of detection.* In the upstream (u.s.) region of detection where $z < 0$, and for any value of $t_1$ such as $0 \leqslant t_1 \leqslant t$, the quantity $(z - Vt_1)$ is negative, so $|z - Vt_1| = Vt_1 - z$ and

$$F^{(\mathrm{u.s.})}(\omega; \rho = 0, z, t) = \frac{1}{V}\int_0^t \frac{e^{wt_1}}{t_1 - z/V} dt_1 = \frac{\exp(wz/V)}{V}\int_{w(z/V)}^{w(z/V-t)} s^{-1}\, e^{-s}\, ds. \tag{45}$$

The second equality in (45) holds upon a change of variable, $s = -w(t_1 - z/V)$. This integral may be expressed in terms of the first Schlömilch function (or exponential integral function) $E_1(\xi)$ (Abramowitz and Stegun 1970). The principal branch of this many-valued function corresponds to any complex argument $\xi$ such as $|\arg \xi| < \pi$. As $z$ is negative, then $\arg z$ may be chosen as either $+\pi$ or $-\pi$. Since $\arg w = -\arctan(2\omega/\nu)$, confining the computation of the many-valued function to its principal determination for convenience, we have to adopt the sign + (−) if the parameter $\omega$ is positive (negative). By means of the identity

$$E_1(\xi) = \int_{\xi}^{+\infty} s^{-1}\, e^{-s}\, ds, \tag{46}$$

with $|\arg \xi| < \pi$, it may be inferred that

$$F^{(\mathrm{u.s.})}(\omega; \rho = 0, z, t) = \frac{e^{\alpha z}}{V}\{E_1(\alpha z) - E_1[\alpha(z - Vt)]\}. \tag{47}$$

The exponential integral function exhibits a logarithmic singularity when its argument tends to zero (Abramowitz and Stegun 1970). It is then clear that the result in (47) becomes an unbounded function at the origin $O$ of the laboratory frame of reference. Within the framework of the antenna in plasma description, only a singularity at the location of the point source is the physically admitted. Generally, results derived from an approach based on the permanent time harmonic regime present singularity at any point lying on the segment $OO'$. Hereafter, a finer analysis of the response of the plasma at this particular region will bring us a physically acceptable model.

We note that the above result may also be inferred from the general expression for $F(\omega; \rho, z, t)$. Indeed, replacing component functions depicted at (41) by their corresponding leading terms deduced from (B.6), the limit of the solution at vanishing $\rho$ is obviously shown to be equal to (47).

*5.2.2. Downstream region of detection.* In the downstream (d.s.) region of detection, $z$ is positive. If this axial coordinate of the detector falls into the condition $t \geqslant (z/V)$, then the distance $|z - Vt_1|$ vanishes at a point inside the interval range of integration. So, the integral (44) is not mathematically defined, only the behaviour of the solution at the vicinity of the $z$-axis may be investigated. The formalism described in the preceding subsection works, however, in the case where $t < (z/V)$. Two separate derivations are thus necessary, according to the relative position of the detector.

We first consider the later situation, that is for earlier time for the evolution: $t < (z/V)$. This region of the streaming plasma is never contaminated by the singularity associated with the antenna point crossing. It is worth noting that in a steady-state description the origin $O'$ of the plasma rest frame is localized at $z = +\infty$. As a consequence, no homologous region can be conformed to this not yet perturbed region, and any corresponding steady-state solution can be put forward.

The plasma response may be derived by use of either the standard limit or the integral definition (44). This yields

$$F^{(\text{d.s.})}(\omega; \rho = 0, z, t) = \frac{\mathrm{e}^{\alpha z}}{V}\{E_1[\alpha(z - Vt)] - E_1(\alpha z)\}. \tag{48}$$

It is seen that the solution at this downstream region of detection spatially oscillates out of the phase of the solution at the upstream region. Instead of a description in which little or nothing is known about the preceding evolution whereby the state is attained, our investigation takes account of precise and complete information about the waves at any time and position.

Now, we consider the situation where $(z - Vt) < 0$. The investigation then concerns the solution at any point which lies between the origins $O$ and $O'$. We adopt the above-mentioned limit formalism. By use of the leading-order terms (B.6) at $\rho \to 0_+$ at the expression of the solution, we obtain

$$F^{(\text{d.s.})}(\omega; \rho \to 0_+, z, t) \simeq -\frac{\mathrm{e}^{\alpha z}}{V}\left\{\left[2\gamma + \ln\left(\frac{1}{2}\alpha\rho\right)\right] \pm \mathrm{i}\pi + E_1(\alpha z) + E_1[\alpha\, \mathrm{e}^{\mp \mathrm{i}\pi}(z - Vt)]\right\}, \tag{49}$$

the upper (lower) sign is taken if $\omega$ is positive (negative). The second form of the solution (49) exhibits a logarithmic divergence at the wake of the point source. This property of the wake electric potential is well admitted in the steady-state regime description. We note that the magnitude of the solution grows indefinitely close to this segment regardless of the rate $\nu$ of the collision. This definitively confirms the assertion that the singularity observed in the value of the wave potential at the transverse directions and close to the origin is ‘frozen’ in the plasma and drifts with the plasma. A further qualitative description of this particular phenomenon of caustic associated with the plasma response will be presented hereafter.

### *5.3. Steady-state limits of the solution*

Equation (41) indicates that the solution is a combination of two component terms. The first term does not depend upon $t$, naturally it contributes itself as an element for the steady-state expression of the solution. Thus, the question is: does the second component term of (41) correspond to a purely transient then vanishing part of the time evolution solution?

At remote time from the switch-on, using equations (B.7), (B.9), (B.14) and (B.15) of appendix B, we have

$$G_1(\rho, z - Vt) \simeq J_0(\alpha\rho) \ln\left(\frac{1}{2}\frac{\rho}{Vt}\right) \tag{50}$$

and

$$G_2(\rho, z - Vt) \simeq -\frac{\pi}{2} Y_0(\alpha\rho) - J_0(\alpha\rho) \ln\left(\frac{1}{2}\frac{\rho}{Vt}\right). \tag{51}$$

This then yields

$$f(\omega;\rho,z-Vt)\simeq-\frac{\pi}{2}Y_0(\alpha\rho), \tag{52}$$

as $t$ tends to $+\infty$. We may conclude that, because the limit of the function $f(\omega;\rho,z-Vt)$ is not zero, we cannot regard it as a complete transient vanishing term. Furthermore, the limit value (52) of the function must be collected as an element for the expression of the steady-state solution to this problem. The exponential prefactor occurring on the right-hand side of (41), when multiplied by $\exp(-\mathrm{i}\omega_0 t)$, leads to the harmonic spatiotemporal variation of the waves. The Doppler effects due to the plasma drift are genuinely introduced by the prefactor exponential term $\exp(\alpha z)$. It is then worth emphasizing that conceptually a steady-state regime arises only when the dissipation is negligibly small, $|\mathrm{Im}\,\alpha|\ll|\mathrm{Re}\,\alpha|$.

## 6. Analysis of the plasma response

The general characteristics of the plasma response subject to the point antenna source situated at the origin of the laboratory frame of reference will be described throughout this section.

### *6.1. Component terms*

We recall that the reduced potential $\chi(\mathbf{r},t)$ defined in (7) is expressed as the combination of two terms involving the complex function $F(\omega;\rho,z,t)$. Hence, the actual electric potential generated by the external charge within the cold moving plasma consists of

$$\phi(\mathbf{r},t)=\phi_{\mathrm{ext}}(\mathbf{r},t)+\exp(-\nu t/2)\chi(\mathbf{r},t). \tag{53}$$

In (53), the external potential is the Coulomb potential given by (11). The influence of the source introduced at a certain point of the undisturbed moving medium is instantaneously felt at all other points of the plasma. This feature turns out to be the consequence of the Galilean electromagnetism approximation which is tantamount to laying on an infinite velocity of light. The second term, $\chi(\mathbf{r},t)$, should be deduced making use of its initial definition in (24), the function $F(\omega,\rho,z,t)$ being given by equation (41) or one of its particular representations, such as (43), (47) and (48). The complete determination of the drifting plasma response subject to an external point source excitation is then performed by this way.

### *6.2. From transients to mature dispersion limit*

The dispersion relation in textbooks and monographs (Delcroix and Bers 1994, Krall and Trivelpiece 1986, Ohnuma 1994, Stix 1992) in general concerns the case of drifting one-dimensional plasma. An insight into the three-dimensional effect to both modes of propagation may be in order here.

Invariably, all the expressions of the functional $F(\omega;\rho,z,t)$ given by either (41), (47), (48) or (49) exhibit the prefactor term in $\exp(\alpha z)$ each. When multiplying this term by $\exp(-\mathrm{i}\omega_0 t)$ which delineates the time periodic behaviour of the antenna excitation, we recover the common harmonic propagation function, $\exp[-\mathrm{i}(\omega_0 t-k_{\mathrm{s,f}}z)]$, describing the temporal and spatial evolution of waves. In addition, the complex wave vectors $k_{\mathrm{s,f}}$ of the both possible modes are readily seen to be

$$k_{\mathrm{s,f}}=\frac{\omega_0\pm\left(\omega_p^2-\nu^2/4\right)^{1/2}}{V}-\mathrm{i}\frac{\nu}{2V}. \tag{54}$$

First, we consider the case where the dissipation remains negligibly small, $\nu \to 0$, equation (54) tends to the expression for the classical dispersion of the space-charge waves. The sign '+' is associated with the slow mode whereas the sign '−' is associated with the fast mode. This prefactor term of harmonic space and time variation function affects instantaneously the plasma response even in the very earlier instant after the switch-on. Therefore, the transient effects on the excited plasma waves are entirely embodied in the expression of their complex amplitudes.

We also emphasize that according to the expression of their harmonic term of propagation, both phase and group velocities of the plasma modes are lying in the direction parallel to the drift velocity **V**. The energy transported by each mode propagates with the plasma. Generally, if an external antenna emanates from a given point of a given medium, some perturbations are generated around this point and wave modes transport the excess of energy away from the region of excitation. For the space-charge cold plasma waves, however, an instantaneous vibration impinges upon the entire plasma, the uniform drift motion converts this temporal perturbation of the medium in propagating wave modes. This point on the description of the excitation and propagation of space-charge plasma waves turns out to be similar to that of the classic one-dimensional case. Here again, it is the space and time distribution of the amplitude of the generated modes which gather the punctual source nature of the excitation and the three-dimensional effects stated for the underlying investigation.

To illustrate the above analysis, we consider the expression of the plasma response along both non-singular parts of the $z$-axis situated at the upstream region and the downstream region of detection. To comply with the condition of the existence of permanent regime and to simplify the formulation the collision frequency $\nu$ is assumed to be zero. In this case, the arguments of each exponential integral function in (47) or (48) are pure imaginary quantities. We thus consider the relations

$$E_1(-\mathrm{i}\xi) = -\mathrm{Ci}(\xi) + \mathrm{i}\left[\tfrac{1}{2}\pi - \mathrm{Si}(\xi)\right], \qquad E_1(+\mathrm{i}\xi) = -\mathrm{Ci}(\xi) - \mathrm{i}\left[\tfrac{1}{2}\pi - \mathrm{Si}(\xi)\right] \tag{55}$$

valid for $|\arg\xi| < \pi/2$, the symbols $\mathrm{Si}(\xi)$ and $\mathrm{Ci}(\xi)$ denoting the sine integral and cosine integral functions (Abramowitz and Stegun 1970) of the argument $\xi$, respectively. Making use of the transformations (55), the derivation of the real part and imaginary part of the function $F(\omega; 0, z, t)$ may be carried out separately. The expressions ensuing from this treatment informally lead to the result

$$\begin{aligned}\chi(\mathbf{r} = z\hat{\mathbf{z}}, t) = -\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p}{2V}H(t)\{&\mathcal{A}_+\cos[\omega_0 t - k_\mathrm{s} z - \Psi_+] \\ &- \mathcal{A}_-\cos[\omega_0 t - k_\mathrm{f} z - \mathrm{sgn}(\omega_-)\Psi_-]\},\end{aligned} \tag{56}$$

where the following notations have been used:

(1) for upstream region, $z < 0$,

$$\mathcal{A}_\pm = \frac{\mathrm{Ci}[|\omega_\pm|(t - |z|/V)] - \mathrm{Ci}[|\omega_\pm||z|/V]}{\cos\Psi_\pm} \tag{57}$$

and

$$\tan\Psi_\pm = \frac{\mathrm{Si}[|\omega_\pm|(t - |z|/V)] - \mathrm{Si}[|\omega_\pm||z|/V]}{\mathrm{Ci}[|\omega_\pm|(t - |z|/V)] - \mathrm{Ci}[|\omega_\pm||z|/V]}; \tag{58}$$

(2) for downstream region, $Vt < z$,

$$\mathcal{A}_\pm = \frac{\mathrm{Ci}[|\omega_\pm|(z/V - t)] - \mathrm{Ci}[|\omega_\pm|z/V]}{\cos\Psi_\pm} \tag{59}$$

and

$$\tan \Psi_{\pm} = \frac{\mathrm{Si}[|\omega_{\pm}|(z/V - t)] - \mathrm{Si}[|\omega_{\pm}|z/V]}{\mathrm{Ci}[|\omega_{\pm}|(z/V - t)] - \mathrm{Ci}[|\omega_{\pm}|z/V]}. \tag{60}$$

In (56) $\mathrm{sgn}(\omega)$ stands for the sign function of $\omega$. At a given position of detection, the transient time rise for the amplitude and the phase of the plasma waves do not depend upon the drift velocity.

At the steady-state limit and in the upstream region of detection ($z < 0$), the amplitude and the additional phase of the wave modes are deduced from the equations

$$\mathcal{A}_{\pm}(\infty) \simeq -\frac{\mathrm{Ci}[|\omega_{\pm}||z|/V]}{\cos \Psi_{\pm}(\infty)} \tag{61}$$

and

$$\tan \Psi_{\pm}(\infty) \simeq \frac{\mathrm{Si}[|\omega_{\pm}||z|/V] - \frac{1}{2}\pi}{\mathrm{Ci}[|\omega_{\pm}||z|/V]}. \tag{62}$$

As expected, the amplitude and the phase evolve with the detection position.

### *6.3. Stability of the plasma modes*

From the foregoing subsection, the essential physical property that the harmonic propagation term displays is the direction of the wave vector. At every point of the plasma, this vector lies along the drift velocity. In some circumstance, it may happen that the collisions are no longer negligible. The full expression (54) for wave vectors has consequently to be considered. The occurrence of the negative imaginary part on both $k_s$ and $k_f$ leads to an explicit spatial damping of the progressive waves. This just mentioned stability condition however concerns the propagation in the downstream region of detection, namely for $z > 0$. We realize that a simple observation of the dispersion relation (54) remains incomplete for an analysis of the stability. Here again, essential information about the actual dynamics of the plasma response can be more clarified and highlighted if a quantitative examination of the space and time variations of both their phase and their amplitude is performed.

First, the magnitude of the excited waves vanishes at remote distance from the source and at the transverse direction, i.e., for a fixed $z$ and $\rho \to \infty$. The stability for a non-drifting fluid plasma is recovered in the transverse direction. Indeed, according to (43) the magnitude of the leading term for the perturbed electric potential vanishes as the Struve function $\mathbf{H}_0(\alpha\rho)$ at a very large distance $\rho$.

Now, we examine the stability along the forward direction of propagation. At a remote distance from the external source of excitation and in the downstream region of detection, the leading asymptotic term of the plasma response may be deduced from (appendix C)

$$F(\omega, \rho, z, t) \simeq \frac{t}{z} J_0(\alpha\rho)\, \mathrm{e}^{\alpha z}, \qquad z \to +\infty. \tag{63}$$

If we multiply (63) by the periodic time variation, $\exp(-\mathrm{i}\omega_0 t)$, we readily find that the complex amplitude of excited mode each evolves in space like $z^{-1}\exp[-(\nu/2V)z]$, as $z \to +\infty$. This confirms the explicit form of the stability which may simply be proffered from the analysis of (54).

Furthermore, the asymptotic development investigated in appendix C shows that at a large negative value of $z$ the plasma response may be deduced from expression (C.10) for the function $F$,

$$F(\omega, \rho, z, t) \simeq \frac{1 - \mathrm{e}^{\alpha V t}}{\alpha V z}, \qquad z \to -\infty.$$

Making use of this expansion, we readily find that the complex amplitude of the excited waves behaves like $(wz)^{-1}\exp(-\mathrm{i}\omega_0 t)$, where $w = (\mathrm{i}\omega_\pm - \nu/2)$. As $z \to -\infty$ and for a fixed time the plasma perturbation goes to zero. Consequently, with regard to the hypothesis on the plasma dissipation we consider here, the plasma response is definitively stable.

### *6.4. Extended singularity*

The underlying plasma wave response is not issued from a variant of the partial differential *wave equation* (Bleistein 1984). Its evolution equation (27) turns out to be a linear first-order equation. Owing to the form of source terms, the solution we have inferred is similar to that of a *simple transport* first-order partial differential equation in a one-dimensional space. The reason is that its spatiotemporal evolution totally stems from the single characteristic coordinate $(z - Vt)$. The general properties of this solution may notably differ from those of the well-known propagating free space electromagnetic waves or those of acoustic waves (Bleistein 1984). Here, an explicit analysis of the excited electric fields singularity may prove useful.

Equation (53) indicates that, on account of the appearance of the Coulomb electric potential $\phi_{\mathrm{ext}}(\mathbf{r}, t)$, the total plasma response exhibits a *source type singularity* at the point $\mathbf{r} = \mathbf{0}$. This is the unique singularity that is physically admitted in a free space electrostatic problem. In addition, it is well known that within the framework of the description of wave excitation in a material medium, the propagating physical quantity may show singularities at surfaces, lines or at isolated points. Here, the exponential integral functions entering the formulae (47) and (48) or the leading-order terms in (49) induce logarithmic singularity along the point source track lying between the origin $O$ of the laboratory frame of reference ($\mathbf{r} = \mathbf{0}$) and the origin $O'$ of the drifting frame of reference ($\mathbf{r} = Vt\hat{\mathbf{z}}$, for any $t > 0$). The wave field exhibits an extended singularity.

The emission of waves by a moving point source whose speed exceeds the wave speed has generally extended singularities. These singularities occur on the envelope of the emitted wave fronts and its cusps, where the waves interfere constructively and thus form caustics. A well-understood example is the emission of acoustic waves by a point source that moves along a straight line with a constant supersonic velocity. In this case a simple caustic forms along a cone issuing from the source, the so-called Mach one, and the wave potential describing the sound amplitude diverges algebraically as this cone is approached from inside. The analogy with the underlying problem comes in the following manner. Galilean time has been chosen for our description. As we have mentioned it above, this implies that $c$, the free space light velocity, tends to infinity, so occurrence of Cherenkov-like emission is physically inconceivable. The usual characteristic speed we can regard is the thermal speed $a$. For a cold plasma investigation, this thermal speed equals zero. We are then here in a case of the extreme limit of the superthermal drift in the sense where the thermal Mach number $M = (V/a) \to +\infty$, whatever the value of $V$. In a drifting warm plasma description (Chee-Seng 1985), the pseudo Mach cone consists of a surface defined by the thermal spherical front $\|\mathbf{r}'\| = at$ and a tangential cone to this thermal sphere and converging at its vertex point at $O$. It is therefore seen that, if $a$ approaches zero the radius of the thermal sphere vanishes and the corresponding Mach cone will regress and fall off into the simple line $OO'$.

We emphasize that the singularity concerns the points lying to the open interval $z \in ]0, Vt[$. The plasma response turns out to be bounded in the front of the 'Mach line' and behind it when we move along the forward direction, i.e., $z$-axis. Indeed, if we calculate the limit of plasma response associated with the reduced electric potential as the detector approaches $O'$

from the downstream region, we obtain (appendix D)

$$\chi(\mathbf{r}'=\mathbf{0}_+,t)=\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p^2}{\Omega}\frac{H(t)}{2V}\left\{\frac{1}{2}\ln\left[\frac{\omega_+^2+\nu^2/2}{\omega_-^2+\nu^2/2}\right]\cos\Omega t\right.$$
$$+\left[\arctan\left(\frac{2\omega_+}{\nu}\right)+\arctan\left(\frac{2\omega_-}{\nu}\right)\right]\sin\Omega t$$
$$\left.+\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\Omega t}E_1\left(\left(\mathrm{i}\omega_+-\frac{\nu}{2}\right)t\right)-\mathrm{e}^{-\mathrm{i}\Omega t}E_1\left(\left(\mathrm{i}\omega_--\frac{\nu}{2}\right)t\right)\right]\right\}, \tag{64}$$

which represents the time evolution of the response. The singularity of the exponential integral at its vanishing argument concerns the imaginary part of the complex response of the plasma (appendix D). Its real part (64) consists of a regular function of the time. On the other hand, if we consider the limit of the same plasma potential field when the detector approaches the origin $O$ from the upstream region, we arrive at (appendix D)

$$\chi(\mathbf{r}=\mathbf{0}_-,t)=-\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p^2}{\Omega}\frac{H(t)\mathrm{e}^{\nu t/2}}{2V}\left\{\frac{1}{2}\ln\left[\frac{\omega_+^2+\nu^2/2}{\omega_-^2+\nu^2/2}\right]\cos\omega_0 t\right.$$
$$-\left[\arctan\left(\frac{2\omega_+}{\nu}\right)-\arctan\left(\frac{2\omega_-}{\nu}\right)\right]\sin\omega_0 t \tag{65}$$
$$\left.+\,\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\omega_0 t}\left\{E_1\left(\left(\frac{\nu}{2}-\mathrm{i}\omega_+\right)t\right)-E_1\left(\left(\frac{\nu}{2}-\mathrm{i}\omega_-\right)t\right)\right\}\right]\right\}.$$

Equation (65) also indicates that the singularity disappears, and the amplitude of the plasma response has a finite limit. For this particular point, we may conclude that the wave potential is singular inside the pseudo Mach envelope, but its amplitude remains regular when one approaches this geometric frontier from outside.

### *6.5. Resonant excitation*

Chassériaux's (1971) description based upon a steady-state regime finds that the drift plasma response to a resonant excitation diverges like $\ln(|\omega_0-\omega_p|)$, as $\omega_0$ tends to $\omega_p$ at all points of the plasma. This clearly indicates that one of the plasma modes acquires a secular behaviour when the impressed frequency is tuned to the plasma frequency. The transient description introduced here turns out to be more suitable to account for such a singular time evolution. As stated by the dispersion relation (54), the fast mode is indeed concerned by a resonance at this frequency. The real part of its wave vector vanishes and the initially progressive wave alters to simple plasma oscillation. Owing to the electric power continuously supplied by the external source the magnitude of the mode oscillation grows without limit.

The plasma response at resonant excitation is derived using the initial integrodifferential representation (19) for the wave solution. We thus take $\nu=0$, $\omega_p=\omega_0=\Omega$. It appears that dynamics of the slow space-charge mode may be depicted by the aid of the formalism described up to now, apart from the fact that $\omega_+=2\omega_p$. About the fast space-charge plasma mode, however, a more developed derivation is required. This yields the result

$$\chi^{(\mathrm{sec})}(\mathbf{r},t)=\frac{q_0}{4\pi\varepsilon_0}\frac{\omega_p}{2V}H(t)h(\rho,z,t)\cos(\omega_p t), \tag{66}$$

where the superscript (sec) emphasizes the secular character of the response and the amplitude evolves in time and space as

$$h(\rho>0,z,t)=\ln|[\rho^2+(z-Vt)^2]^{1/2}-(z-Vt)|-\ln|(\rho^2+z^2)^{1/2}-z| \tag{67}$$

and

$$h(\rho = 0, z, t) = \ln(1 - Vt/z), \qquad \text{if} \quad z < 0 \text{ or } Vt < z. \tag{68}$$

Unlike the problem of cold stationary plasma resonant oscillation for which the secular term grows linearly with $t$ (Randriamboarison 1997), here the time rise of the resonant fast mode is seen to be in $\ln(t)$ at elapsed instant from the switch-on. The amplitude of the plasma response then grows indefinitely in time. If any mathematical form of this amplitude may be deduced at a time $t$ from the steady-state treatment of the problem, the present investigation provides us with the exact and complete description of the temporal evolution of both modes.

## 7. Summary and conclusions

The formalism introduced here provides a complete description for the space-charge waves emanated by a punctual antenna immersed within a cold drifting plasma. Indeed, a technique based on the direct operational methods allowed us to transform the governing equation to a more suitable and tractable one for an explicit resolution of the problem. The condition for a steady-state modal approximation for the waves exists is that the dissipation stays at a very negligible level. Here, this restriction was not imposed on the value of the collision frequency. The exact development is more suited for the analysis of the excitation and propagation of the generated waves.

The plasma responses were expressed in terms of Bessel and Neumann functions and series of the Lommel functions. The algebraic expressions for the solution present the advantage that they describe a more general situation, so analytical investigations at some particular boundaries or in some limiting cases may be readily deduced from them. Throughout this work, the opportunity has been taken to get some more understanding of the dynamics of the well-known space-charge plasma modes.

Namely, our result confirms the appearance of an extended singularity situated along the wake of the antenna within the drifting cold plasma (Chassériaux 1971, Mourgues *et al* 1980). From a physical point of view, this singularity has been interpreted as a caustic built up by the external charge source displacement throughout the medium. The existence of unbounded solution along a particular geometric line stems its mathematical origin for a propagation of singular data along a characteristic base curve of the differential equation. For the sake of simplicity, however, a very idealized model was investigated: (1) a cold plasma approach; (2) a punctual antenna for the excitation source. We note that a more realistic model takes into account temperature effects. Moreover, antennas immersed in the moving plasma can never be pointlike. In general, they have finite extent in space.

In the classical steady-state approach, the effects of a small temperature of plasma particles remove the divergences of the potential on the axis on which the external source moves (Fiala 1973, Michel 1976, Mourgues *et al* 1980). The thermal speed of the plasma steps in as a new physical parameter for the description of the problem. According to the relative value of this speed with respect to the plasma drift velocity, two different cases hold: the subthermal drift and the superthermal drift. The description of the space-charge plasma modes undergoes some drastic transformations in both situations. Namely, for the later mentioned case, the line caustic derived here in the cold plasma model becomes a full extended Mach or Cherenkov cone. It is expected that this alteration of the caustic wake potential also holds in the case of impulsive or transient excitation of a drifting warm plasma. This analysis will be carried out in a subsequent work.

Furthermore, any finite sized antenna can be seen as a juxtaposition of separated point charges; the field potential due to a finite sized charge will be contributed due to the interference of potential excited by the individual point particles. The destructive interference of the phases due to individual point charges that are separated by small but finite distances leads to a reshaping of the caustic. The problem of a finite extent antenna is related to that investigated numerically by Melandso and Goree (1995) or Bose and Janaki (2005). In the physical situation investigated here, the response of the cold plasma appears to be unidirectional waves propagating along the drift velocity. The interference mechanism of the unidirectional wave may diverge from that of omnidirectional classical waves. In some specific applications, the numerical investigation of the finite size source effects on the cold plasma space-charge modes remains to be done.

The slow space-charge wave has the property that it carries negative stored energy. This means that disturbances grow as the wave gives up energy. This effect is what causes the class of beam-plasma instabilities and makes possible a wide variety of microwave amplifiers and oscillators such as klystrons. Generally, the instability appears if the dissipation of the background plasma which supports the beam may not be discarded (Delcroix and Bers 1994, for example). We showed here that if the collisions characterize the drifting plasma itself, both space-charge modes evolve like common stable waves.

Finally, the opportunity has also been taken to describe the resonant excitation of the cold drifting plasma modes. Because $\omega_0 = \omega_p$ constitutes a regular excitation for the slow mode, the corresponding generated wave behaves normally. Thus, its wave field may be deduced by replacing the excitation frequency by the plasma frequency. The plasma resonant behaviour however fully bears upon the fast space-charge wave. From an initial progressive wave, the later mode lapses to simple oscillations. The amplitude of its generated temporal vibrations then grows and diverges logarithmically at elapsed time from the switch-on.

## Acknowledgments

I would like to thank Daniel Mas for a careful reading of the manuscript. This research was supported by the Département des Sciences de l'Univers (SDU) and the Institut des Sciences de l'Univers (INSU) in France.

## Appendix A. Derivation of the integral $A_k(z)$

In this appendix, a hint on the derivation of the integral $A_k(z)$ leading to equations (32) and (33) is given.

In view of (29) and (31), it is identified that

$$A_k(z) = \int_0^z \frac{\zeta^k}{(\rho^2+\zeta^2)^{1/2}}\,\mathrm{d}\zeta, \qquad k = 0, 1, 2, \ldots. \tag{A.1}$$

First, we have

$$\int \frac{\mathrm{d}x}{(a^2+x^2)^{1/2}} = \ln[x + (a^2+x^2)^{1/2}] + C_0. \tag{A.2}$$

Then, the following integral identities hold (Gröbner and Hofreiter 1965),

$$\int \frac{x^k}{(a^2+x^2)^{1/2}}\,\mathrm{d}x = \frac{1}{2}(a^2+x^2)^{1/2}\sum_{\mu=0}^{\frac{k}{2}-1}(-1)^\mu \frac{\left(\frac{k}{2}-\mu+\frac{1}{2}\right)_\mu}{\left(\frac{k}{2}-\mu+1\right)_{\mu+1}} a^{2\mu}x^{k-2\mu-1}$$
$$+ (-1)^{\frac{k}{2}}\frac{(1/2)_{\frac{k}{2}}}{(1)_{\frac{k}{2}}} a^k \ln[x + (a^2+x^2)^{1/2}] + C_1, \tag{A.3}$$

for odd values of $k$, and

$$\int \frac{x^k}{(a^2+x^2)^{1/2}}\,\mathrm{d}x = \frac{1}{2}(a^2+x^2)^{1/2}\sum_{\mu=0}^{\frac{k-1}{2}}(-1)^\mu \frac{\left(\frac{k}{2}-\mu+\frac{1}{2}\right)_\mu}{\left(\frac{k}{2}-\mu+1\right)_{\mu+1}} a^{2\mu}x^{k-2\mu-1} + C_2, \tag{A.4}$$

for even values of $k$. Instead of initial Kramp's symbols of the cited reference, Pochhammer's symbols $(\alpha)_\mu = \alpha(\alpha+1)\cdots(\alpha+\mu-1)$ have been used in the expression of (A.3) and (A.4). Upon replacing the index $k$ by $2j$ and $(2j+1)$ in equations (A.3) and (A.4), respectively, $j = 0, 1, 2, \ldots,$ and applying the above formula to calculate definite integrals from (A.1), we readily find the results (32) and (33).

## Appendix B. Leading-order terms for series and asymptotic representations

We derive the leading terms of the series expansions and asymptotic forms of the function $f(\omega; \rho, z)$ defined by equation (42).

The underlying function is the summation of two component terms. As $0 \leqslant \rho \ll 1$, the first one behaves as

$$G_1(\rho, z) = \ln\left(\frac{1}{2}\frac{\rho}{|z|}\right) + O(\rho). \tag{B.1}$$

Thus, according to the series representation of Lommel functions (Watson 1938, Luke 1962), we have

$$s_{k,0}(\alpha\rho) = \frac{(\alpha\rho)^{k+1}}{k+1} + O(\rho^{k+3}), \tag{B.2}$$

for $0 \leqslant \rho \ll 1$. Hence,

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\left(\frac{z}{\rho}\right)^k s_{k,0}(\alpha\rho) \simeq -\frac{\rho}{z}\sum_{m=1}^{+\infty}\frac{(-1)^m}{m!}\frac{(\alpha z)^m}{m}. \tag{B.3}$$

Using the series expansion of the exponential integral or Schlömilch's function of order 1 and argument $\xi$ (Abramowitz and Stegun 1970)

$$E_1(\xi) = -\gamma - \ln\xi - \sum_{m=1}^{+\infty}\frac{(-1)^m}{m!}\frac{(\xi)^m}{m}, \qquad |\arg\xi| < \pi, \tag{B.4}$$

where the symbol $\gamma$ denotes the Euler–Mascheroni constant, we find

$$G_2(\rho, z) = \gamma + \ln(\alpha z) + E_1(\alpha z) + O(\rho^2), \tag{B.5}$$

provided $|\arg(\alpha z)| < \pi$. The sum of (B.1) and (B.5) yields

$$f(\omega; \rho, z) = (\gamma \pm \mathrm{i}\pi) + \ln(\alpha\rho/2) + E_1(\alpha z) + O(\rho), \qquad \rho \to 0_+ \tag{B.6}$$

where the upper (lower) sign is chosen if the quantity $\mathrm{Im}(\alpha)$ is less (great) than zero.

Now, our attention is paid to the large-$t$ behaviour of the function $f(\omega; \rho, z - Vt)$. This limit is important in the sense that it provides us with the solution at the steady-state regime. It turns out that the first component $G_1(\rho, z - Vt)$ may be kept in the intermediate form of

$$G_1(\rho, z - Vt) = J_0(\alpha\rho)\ln\left[(1 + V^2t^2/\rho^2)^{1/2} - Vt/\rho\right], \tag{B.7}$$

for the derivation of the asymptotic leading as $t \to +\infty$. When combining with a component term of $G_2(\rho, z - Vt)$ derived hereafter the contribution of (B.7) to the required asymptotic estimation will finally vanish.

The Lommel function may be written in its integral representation (Watson 1938, Luke 1962)

$$s_{k,0}(\xi) = \frac{\pi}{2}\left[Y_0(\xi)\int_0^{\xi}\tau^k J_0(\tau)\,\mathrm{d}\tau - J_0(\xi)\int_0^{\xi}\tau^k Y_0(\tau)\,\mathrm{d}\tau\right], \tag{B.8}$$

with $Y_0$ designating the Neumann function of order 0. From this, we may put the second component function of $f(\omega; \rho, z - Vt)$ in the alternative form of

$$G_2(\rho, z - Vt) = -\frac{\pi}{2}[1 + (z - Vt)^2/\rho^2]^{1/2}[Y_0(\alpha\rho)\mathcal{L}_1(t, z) - J_0(\alpha\rho)\mathcal{L}_2(t, z)], \tag{B.9}$$

with

$$\mathcal{L}_1(t) = \int_0^{\alpha\rho}\exp\left[\frac{Vt - z}{\rho}\tau\right]J_0(\tau)\,\mathrm{d}\tau, \tag{B.10}$$

and

$$\mathcal{L}_2(t) = \int_0^{\alpha\rho}\exp\left[\frac{Vt - z}{\rho}\tau\right]Y_0(\tau)\,\mathrm{d}\tau. \tag{B.11}$$

If $t \gg 1$, these integrals can be put in the form of

$$\mathcal{L}_1(t) \simeq (\alpha\rho)\int_0^1 \mathrm{e}^{-\lambda\sigma}J_0(\alpha\rho\sigma)\,\mathrm{d}\sigma \tag{B.12}$$

and

$$\mathcal{L}_2(t) \simeq (\alpha\rho)\int_0^1 \mathrm{e}^{-\lambda\sigma}Y_0(\alpha\rho\sigma)\,\mathrm{d}\sigma, \tag{B.13}$$

where $\lambda \equiv -wt$, $|\lambda| \to +\infty$. By means of the standard procedure for asymptotic integral approximation, based on a combination of the Laplace method followed by Watson's lemma (e.g., Bender and Orszag (1978), Bleistein and Handelsman (1986)), it can be shown that only exponentially small errors are introduced if we approximate these integrals by the Laplace transform of the Bessel and the Neumann functions, respectively. Performing their integration (Roberts and Kaufman 1966), the required asymptotic series converge to

$$\mathcal{L}_1(t) \simeq (1 + V^2t^2/\rho^2)^{-1/2} \tag{B.14}$$

and

$$\mathcal{L}_2(t) \simeq -\frac{2}{\pi}(1 + V^2t^2/\rho^2)^{-1/2}\ln[(1 + V^2t^2/\rho^2)^{1/2} - Vt/\rho]. \tag{B.15}$$

## Appendix C. Asymptotic forms of the function $F(\omega, \rho, z, t)$

In order to check the stability of the solution, the asymptotic leading terms of the function $F(\omega, \rho, z, t)$ at $z \to \pm\infty$ are required. We note that this function is obtained by the combination expressions of $f(\omega; \rho, z)$ being studied in the foregoing appendix. Hence, for $z \to +\infty$ a formalism similar to that was developed in appendix B provides us with the leading terms for the required asymptotic representation. The following results ensue:

$$G_1(\rho, z) = J_0(\alpha\rho)\ln\left[2\frac{z}{\rho} + O(z^{-1})\right] \tag{C.1}$$

and

$$G_2(\rho, z) = \frac{\pi}{2}\frac{\rho}{z}\mathrm{e}^{-\alpha z}[J_0(\alpha\rho)Y_1(\alpha\rho) - J_1(\alpha\rho)Y_0(\alpha\rho)] + O(z^{-1}). \tag{C.2}$$

Plugging (C.1) and (C.2) into equation (41) which characterizes the function $F$, we find

$$F(\omega, \rho, z, t) \simeq \frac{t}{z} J_0(\alpha\rho)\, e^{\alpha z}, \qquad z \to +\infty. \tag{C.3}$$

We note that this function is subject to the so-called Stokes phenomenon. Its asymptotic form in the sector where $z$ goes to $-\infty$ differs from that of the sector where $z$ tends to $+\infty$. One should thus develop an alternative and independent treatment in order to handle the former limit.

We start from the initial integral (29). That is

$$\mathcal{I}(z) = \int_0^z \frac{\exp(-\alpha\zeta)}{(\rho^2 + \zeta^2)^{1/2}}\, d\zeta. \tag{C.4}$$

Since $\mathrm{Re}\,\alpha < 0$, the integral remains bounded at the limit $z \to -\infty$. The integration may then be linearly decomposed into a sum and making use a change of variable $\xi = -\zeta$, this yields $\mathcal{I}(z) = \mathcal{I}_0 + \mathcal{I}_1(z)$ with

$$\mathcal{I}_0 = -\int_0^{+\infty} \frac{\exp(\alpha\xi)}{(\rho^2 + \xi^2)^{1/2}}\, d\xi \tag{C.5}$$

and

$$\mathcal{I}_1(z) = \int_{|z|}^{+\infty} \frac{\exp(\alpha\xi)}{(\rho^2 + \xi^2)^{1/2}}\, d\xi. \tag{C.6}$$

The integral $(-\mathcal{I}_0)$ appearing in (C.5) represents the Laplace transform of $u(\xi) = (\rho^2 + \xi^2)^{-1/2}$, evaluated at the point $s = -\alpha$. So one has

$$\mathcal{I}_0 = \frac{\pi}{2}[Y_0(\alpha\rho) - \mathbf{E}_0(\alpha\rho)], \tag{C.7}$$

where $\mathbf{E}_0$ stands for the Weber–Lommel function of order 0 (Jahnke *et al* 1960, Abramowitz and Stegun 1970).

Now, if $\xi > \rho$, then

$$(\rho^2 + \xi^2)^{-1/2} = \frac{1}{\xi} \sum_{k=0}^{+\infty} \mathrm{C}^k_{-1/2} (\rho/\xi)^{2k}. \tag{C.8}$$

Substituting expansion (C.8) into (C.6), we show that the integral $\mathcal{I}_1(z)$ may be put as a series of Schlömilch functions $E_k(x)$ (Abramowitz and Stegun 1970). That is,

$$\mathcal{I}_1(z) = \sum_{k=0}^{+\infty} \mathrm{C}^k_{-1/2} \left(\frac{\rho}{z}\right)^{2k} E_{2k+1}(\alpha z), \tag{C.9}$$

provided $-z = |z| > \rho$ and $|\mathrm{Arg}(\alpha z)| < \pi$. Assuming that $|\mathrm{Arg}(\zeta)| < 3\pi/2$, the asymptotic leading term for the Schlömilch function each rises from the formula $E_k(\zeta) = (e^{-\zeta}/\zeta)[1 + (k/\zeta) + O(\zeta^{-2})]$, $|\zeta| \gg 1$. Then, if we conform with equation (28), this leads to the result

$$F(\omega, \rho, z, t) \simeq \frac{1 - e^{\alpha V t}}{\alpha V z}, \qquad z \to -\infty. \tag{C.10}$$

Equations (C.3) and (C.10) eventually indicates that the function $F(\omega, \rho, z, t)$ decreases as $(1/z)$ at both limits $z \to +\infty$ and $z \to -\infty$ and for a fixed value of $\rho$ and $t$.

## Appendix D. Calculation of the time course of $\chi(\mathbf{r}, t)$ at $O$ and $O'$

Equation (24) is equivalent to

$$\chi(\mathbf{r}, t) = -\frac{q_0}{4\pi\varepsilon_0} \frac{\omega_p^2}{\Omega} \frac{1}{2} H(t)\, \mathrm{Re}[g(\omega_0; \rho, z, t)], \tag{D.1}$$

with the notation

$$g(\omega_0;\rho,z,t)=[F(\omega_+;\rho,z,t)-F(\omega_-;\rho,z,t)]\,\mathrm{e}^{(\nu/2-i\omega_0)t}. \tag{D.2}$$

We emphasize that the time course of the plasma response is fully characterized by the real function $\mathrm{Re}[g(\omega_0;\rho,z,t)]$ visible on the right-hand side of (D.1).

First, at any positive time $t$ and in the downstream region of detection where $(z>Vt)$, we have

$$\begin{aligned}g(\omega_0;0,z,t)={}&V^{-1}\exp[(\mathrm{i}\omega_0-\nu/2)(z/V-t)]\{\mathrm{e}^{\mathrm{i}\Omega z/V}[E_1((\mathrm{i}\omega_+-\nu/2)(z/V-t))\\&-E_1((\mathrm{i}\omega_+-\nu/2)(z/V))]-\mathrm{e}^{-\mathrm{i}\Omega z/V}[E_1((\mathrm{i}\omega_--\nu/2)(z/V-t))\\&-E_1((\mathrm{i}\omega_--\nu/2)(z/V))]\}.\end{aligned} \tag{D.3}$$

The above expression has been written using (48), the limit of the function $F$ when $\rho$ equals zero. We regard the time course of the wave potential at the origin $O'$ of the reference frame of the moving plasma. This corresponds to the case where, for a fixed time $t$, the variable $z$ tends to $Vt$. As $|t-z/V|\ll 1$, then the prefactor term $\exp[(\mathrm{i}\omega_0-\nu/2)(z/V-t)]\simeq 1$. In addition, according to (B.4), the leading-order terms for the exponential integral function at small argument ($|\xi|\ll 1$) are deduced from $E_1(\xi)\simeq-\gamma-\ln\xi+O(\xi)$, $|\arg\xi|<\pi$. Now, if we set

$$\mathcal{D}_{O'}\equiv\mathrm{e}^{\mathrm{i}\Omega t}E_1[(\mathrm{i}\omega_+-\nu/2)(z/V-t)]-\mathrm{e}^{-\mathrm{i}\Omega t}E_1[(\mathrm{i}\omega_--\nu/2)(z/V-t)] \tag{D.4}$$

then, in the case where $|t-z/V|$ is close to zero,

$$\begin{aligned}\mathcal{D}_{O'}\simeq{}&-2\mathrm{i}[\gamma+\ln(z/V-t)]\sin\Omega t-(1/2)\,\mathrm{e}^{\mathrm{i}\Omega t}\left[\ln\left(\omega_+^2+\nu^2/4\right)-2\mathrm{i}\arctan(2\omega_+/\nu)\right]\\&+(1/2)\,\mathrm{e}^{-\mathrm{i}\Omega t}[\ln(\omega_-^2+\nu^2/4)-2\,\mathrm{i}\arctan(2\omega_-/\nu)]\\\simeq{}&\frac{1}{2}\ln\left(\frac{\omega_-^2+\nu^2/4}{\omega_+^2+\nu^2/4}\right)\cos\Omega t-\left[\arctan\left(\frac{2\omega_+}{\nu}\right)+\arctan\left(\frac{2\omega_-}{\nu}\right)\right]\sin\Omega t\\&-2\mathrm{i}[\gamma+\ln(z/V-t)]\sin\Omega t-(1/2)\mathrm{i}\ln\left[\left(\omega_+^2+\nu^2/4\right)\left(\omega_-^2+\nu^2/4\right)\right]\sin\Omega t\\&+\mathrm{i}\left[\arctan\left(\frac{2\omega_+}{\nu}\right)-\arctan\left(\frac{2\omega_-}{\nu}\right)\right]\cos\Omega t.\end{aligned} \tag{D.5}$$

These equations clearly indicate that the singular term $\ln(z/V-t)$ is apparent in the imaginary part of $\mathcal{D}_{O'}$. It does not concern the real part, $\mathrm{Re}[g(\omega_0;0,z,t)]$. Hence, close to the origin $O'$ but along the forward direction in the downstream region from the source and its wake, the time course of the wave potential reads

$$\begin{aligned}\mathrm{Re}[g(\omega_0;0,z=Vt,t)]={}&-V^{-1}\mathrm{Re}\{\mathrm{e}^{i\Omega t}E_1[(\mathrm{i}\omega_+-\nu/2)t]-\mathrm{e}^{-\mathrm{i}\Omega t}E_1[(\mathrm{i}\omega_--\nu/2)t]\}\\&+V^{-1}\frac{1}{2}\ln\left(\frac{\omega_-^2+\nu^2/4}{\omega_+^2+\nu^2/4}\right)\cos(\Omega t)\\&-V^{-1}[\arctan(2\omega_+/\nu)-\arctan(2\omega_-/\nu)]\sin(\Omega t).\end{aligned} \tag{D.6}$$

The full equation (64) for the plasma response is obtained when we plug (D.6) into (D.1).

Second, in the same way as in the foregoing development, we examine the limit of the wave electric potential when the space variable $z$ tends to zero whereas $(z<0)$. We write

$$\begin{aligned}g(\omega_0;0,z\to 0,t)\simeq{}&V^{-1}\exp[(\nu/2-\mathrm{i}\omega_0)t]\{E_1[(\nu/2-\mathrm{i}\omega_+)t]-E_1[(\nu/2-\mathrm{i}\omega_-)t]\\&+E_1[(i\omega_--\nu/2)z/V]-E_1[(\mathrm{i}\omega_+-\nu/2)z/V]\}.\end{aligned} \tag{D.7}$$

Let $\mathcal{D}_O$ define the $z$-dependent function

$$\mathcal{D}_O\equiv E_1[(\mathrm{i}\omega_--\nu/2)z/V]-E_1[(\mathrm{i}\omega_+-\nu/2)z/V]. \tag{D.8}$$

When expanded around $z = 0_-$ this quantity becomes

$$\mathcal{D}_O \simeq -\ln(\mathrm{i}\omega_- - \nu/2) + \ln(\mathrm{i}\omega_+ - \nu/2)$$
$$\simeq \frac{1}{2}\ln\left(\frac{\omega_+^2 + \nu^2/4}{\omega_-^2 + \nu^2/4}\right) - \mathrm{i}[\arctan(2\omega_+/\nu) - \arctan(2\omega_-/\nu)]. \quad \text{(D.9)}$$

The singular terms immediately vanish and (D.9) yields a regular function. Hence, at the origin $O$ of the laboratory frame of reference ($z = 0_-$), the electric potential evolves in time like

$$\mathrm{Re}[g(\omega_0; 0, z = 0, t)] = V^{-1}\,\mathrm{e}^{\nu t/2}\,\mathrm{Re}\{\mathrm{e}^{-\mathrm{i}\omega_0 t}(E_1[(\nu/2 - \mathrm{i}\omega_+)t] - E_1[(\nu/2 - \mathrm{i}\omega_-)t])\}$$
$$+ V^{-1}\,\mathrm{e}^{\nu t/2}\frac{1}{2}\ln\left(\frac{\omega_+^2 + \nu^2/4}{\omega_-^2 + \nu^2/4}\right)\cos(\omega_0 t) - V^{-1}\,\mathrm{e}^{\nu t/2}[\arctan(2\omega_+/\nu)$$
$$- \arctan(2\omega_-/\nu)]\sin(\omega_0 t). \quad \text{(D.10)}$$

Introducing (D.10) in (D.1) we arrive at the full expression (65) of the plasma response.

## References

Abramowitz M and Stegun I 1970 *Handbook of Mathematical Functions* 9th edn (New York: Dover)
Arfken G 1970 *Mathematical methods for Physicists* 2nd edn (New York: Academic)
Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)
Bleistein N 1984 *Mathematical Methods for Wave Phenomena* (Orlando, FL: Academic)
Bleistein N and Handelsman R A 1986 *Asymptotic Expansions of Integrals* (New York: Dover)
Bose A and Janaki M S 2005 *Phys. Plasmas* **12** 052105 1–6
Chassériaux J -M 1971 *Suppl. J. Phys.* **32** C5 B 195
Chawla B R and Unz H 1962 *Radio Sci.* **1** 1055
Chee-Seng L 1985 *J. Plasma Phys.* **33** 83
de Montigny M and Rousseaux G 2006 *Eur. J. Phys.* **27** 755
Delcroix J-L and Bers A 1994 *Physique des plasmas* vol 1 (Paris: InterEditions/CNRS)
Dunn H S 1967 *Proc. Camb. Phil. Soc.* **63** 155
Fiala V 1973 *J. Plasma Phys.* **10** 371
Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* (New York: Academic)
Gröbner W and Hofreiter N 1965 *Integraltafel I: Unbestimmte Integrale* (Wien: Springer)
Jahnke E, Emde F and Lösch F 1960 *Tables of Higher Functions* 6th edn (New York: McGraw-Hill)
Kaplan W 1967 *Ordinary Differential Equations* 5th edn (Reading, MA: Addison-Wesley) p 164
Krall N A and Trivelpiece A W 1986 *Principles of Plasma Physics* (San Francisco: San Francisco Press)
Le Bellac M and Lévy-Leblond J-M 1973 *Nuovo Cimento* B **14** 217
Lindell I V 2000 *Prog. Electromagn. Res.* **26** 293
Luke Y L 1962 *Integrals of Bessel Functions* (New York: McGraw-Hill)
Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for Special Functions of Mathematical Physics* 3rd edn (Berlin: Springer)
Melandso F and Goree J 1995 *Phys. Rev.* E **52** 5312
Michel E 1976 *J. Plasma Phys.* **15** 395
Mourgues G, Fijalkow E and Feix M R 1980 *Plasma Phys.* **22** 367
Ng E W 1968 *J. Comput. Phys.* **3** 334
Ohnuma T 1994 *Radiation Phenomena in Plasmas* (Singapore: World Scientific)
Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 *Numerical Recipes* 2nd edn (Cambridge: Cambridge University Press)
Randriamboarison O C 1997 *Phys. Plasmas* **4** 2336
Roberts G E and Kaufman H 1966 *Table of Laplace Transforms* (Philadelphia, PA: Saunders)
Spanier J and Oldham K B 1987 *An Atlas of Functions* (New York: Hemisphere Publishing Corporation)
Stix T H 1992 *Waves in Plasmas* (New York: American Institute of Physics)
Swanson D G 1989 *Plasma waves* (Boston, MA: Academic)
Unz H 1966 *Phys. Rev.* **146** 92
Vaidya A and Farina C 1991 *Phys. Lett.* A **153** 265
Watson N G 1938 *A Treatise on the Theory of Bessel Functions* 2nd edn (Cambridge: Cambridge University Press)